Parallel mesh generation

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Abstract. The efficient parallelisation of the finite element method is based on non-overlapping partitioning of the computational domain into an appropriate number of subdaomins. The problem size required for efficient application of parallel solution techniques is considerably large. The problem description in terms of finite element nodes and elements is complicated and difficult to handle with respect to the required main memory and file size. We describe a parallel solution method to perform mesh partitioning without prior mesh generation. The geometric description of the computational domain consists of vertices, edges and faces and boundary conditions, loads and mesh density parameters. The geometric description is recursively partitioned. The domain interfaces are minimized with respect to the number of coupling finite element nodes. Load balance is ensured for graded and locally refined meshes. Applications for two-dimensional models in structural mechanics are demonstrated.

1 Introduction

Mesh generation is a well-known hard problem in finite element method. The finite element method allows modeling of physical phenomena on unstructured, graded meshes in contrast to finite difference method using structured or block-structured computational meshes. Usability and flexibility of finite element method is proved in a vast amount of applications. Usually, the input data for the unstructured meshes result from CAD data.
The need to generate finite element meshes quickly is a common requirement of most computational fields. This is especially true for adaptive finite element processes. Therefore, the need for parallel mesh generation is well justified. Parallel solution techniques are mainly based on partitioning of the computational data and mapping them onto an particular parallel machine. For distributed memory parallel machines geometrically motivated partitioning is very efficient to construct fast solvers. Additionally parallel mesh generation avoids the bottleneck of serial mesh generation arising from storing, partitioning and loading large computational meshes.

The partitioning problem is known to be NP-complete (proof see [21]). That means the problem cannot be solved in polynomial time. Parallelising the partitioning approach reduces significantly the time for partitioning a given mesh. The mapping problem is known to be NP-complete, too. Although, usefull heuristic approaches are available (see [5], [10]).
A large number of research groups works in the field of parallel mesh generation [30], [22], [13], [23], [15], [28], [31]. The usual attempt to generate and partition a given problem in parallel is made in two steps using a so-called background mesh.

Step 1 Solve the partitioning problem for the background mesh.

Step 2 Refine the background mesh to achieve the required mesh density.

Step 1 is usually performed in a recursive manner resulting in a number of idle processors during the first recursion levels. Step 2 is inherently parallel.
The motivation of the development of these algorithms is twofold:

1. The size of the partitioning problem for the background mesh is significantly reduced compared to the original problem size resulting in a significant shorter runtime.

2. The mesh refinement process is parallelized using the (geometric) locality.

The generated partitioning should ensure connected and well-shaped subdomains on every processor and equally distributed numbers of elements or nodes for the non-overlapping domain decomposition approach.
The mapping problem is usually not solved by the approaches in the techniques referenced. But the parallel generation of the mesh is motivated by the elimination of the mesh loading and distribution process. Re-assigning the mesh to a more appropriate mapping is contra productive. Therefore, the generated partitioning should also ensure a near optimal mapping.
The strategies based on background meshes are dual to the multilevel methods developed mainly for the high quality partitioning algorithms based on the spectral bisection method. Origins of this method can be found in the works of Barnes and Hoffman [3] and it was first proposed by Pothen [25] for improving graph ordering for sparse matrix factorisation. The recursive partitioning is well suited for a near optimal mapping of the partitioning to hypercube and mesh topologies.
Although spectral bisection produces best quality decompositions (compare [26], [29], [20]) the complexity of the numerical Eigenvalue problem to be solved is significant. Therefore, multilevel methods reduce the Eigenvalue problem by gathering elements of the fine computational mesh to larger elements. The partitioning is performed for the smaller problem with the larger elements (see [2], [14], [17]). Jones and Plassman [16], Barnard and Simon [1], Roose and Driesche [8], [7], [9] are proposing parallel versions of the algorithm to reduce runtime.

Nevertheless, the quality of the final partitioning of the refined mesh has to be improved by local techniques to minimize cut length and optimize the subdomain shape. Techniques applicable both for serial and parallel improvements are Kernighan-Lin heuristic [18] or Helpful-set strategy [6].
In contrast to the background mesh approach we propose a method for parallel mesh generation working directly on the CAD input data and without a predefined background mesh. Partitioning cuts are introduced as additional geometric constraints to the original data. The quality of the partitioning is directly ensured by straight cut lines. The resulting subdomains are well shaped. Final improvement by a local heuristic is not necessary. The generated partitions are nearly optimal mapped to the underlying hardware topology. The parallel mesh generation algorithm is described in section 2. Results obtained for unstructured not graded and graded meshes are discussed in section 3.

2 Parallel Mesh Generation

In the following subsection we describe the input data for the proposed meshing algorithm. The inertial method is then described briefly. We explain more detailed aspects of the subdomain boundary generation in the following two subsections.

2.1 Input data

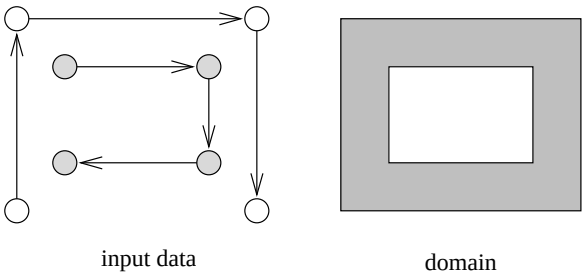

Fig. 1. *Input data of mesh generator*

The two-dimensional geometry is described by linear primitives. These data origin from a CAD description and consists of points, lines and polygons describing the outline and holes of a two-dimensional computational domain, properties assigned to subdomains, polygons (or boundaries of subdomains), lines (segments of polygons) and single points (see Figure 1).
Points are assigned information on the expected finite element size. Expected element size is interpolated between known points. Lines are defined as linear connections of previously defined points. Polygons consist of an ordered set of

lines. Polygons are assumed to be closed. One of the polygons describes the outer boundary of the two-dimensional region. Holes are described by special tags to polygons. Additionally, lines and points can reflect lines or points in the computational model.

2.2 Inertial method

The main idea for the partitioning origins from the inertial method, for example [11]. The domain is split into two equally weighted subdomains along the axis of the larger inertial momentum. The quality of the cut is problem dependend. The inertial mehtod is applied to pre-existing meshing the co-ordinate information. Our method is directly based on the partitioning of the CAD geometry. Therefore, cut lines are added to the geometry description of the CAD data and extend the input data of the meshing algorihm.

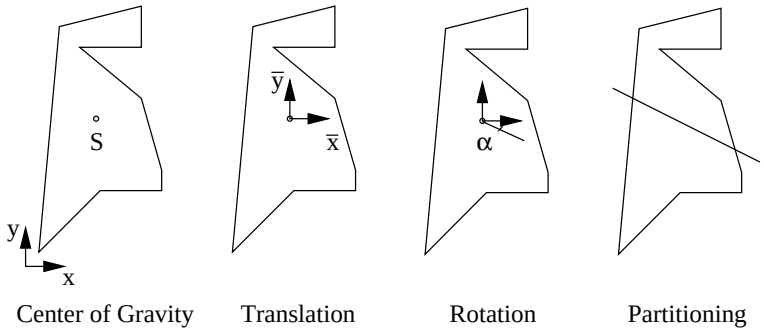

Fig. 2. *Recursive bipartition by inertial method*

The mesh density is governed by a pointwise defined density function $f(x, y)$. Usually, such a function defines the requested element size. This element size defines the number of elements in the surrounding of these points. To ensure equally distributed computational load in the mesh generation and in the finite element computation the number of elements should be the same on every processor. Therefore, we construct from the function describing the element size a function describing the computational load.
We are using in the two-dimensional case for an expected element size w the function $f(x, y) = \frac{1}{w^2}$ measuring the element maximum diagonal length or the surrounding circle, respectively. Alternatively, we are using the function $f(x, y) = (\bar{w} - w)^2$ with $\bar{w}$ indicating a mean or maximum element side length. Both functions are well suited for a weighting function applied to the domains describing the problem to be used in the inertial method.
One recursion step of bipartition is performed as follows (see Figure 2). Initially, the center of gravity of the polygonial bounded geometry is calculated by numerical integration with

$$x_s = \frac{\int x \cdot f(x,y)dA}{\int f(x,y)dA}, \quad y_s = \frac{\int y \cdot f(x,y)dA}{\int f(x,y)dA} \tag{1}$$

and the problem is translated into the $\bar{x}, \bar{y}$ co-ordinate system. The inertial momentum are computed by

$$I_{\bar{x}} = \int \bar{y}^2 \cdot f(\bar{x},\bar{y})dA, \quad I_{\bar{y}} = \int \bar{x}^2 \cdot f(\bar{x},\bar{y})dA$$
$$I_{\bar{x}\bar{y}} = -\int \bar{x} \cdot \bar{y} \cdot f(\bar{x},\bar{y})dA \tag{2}$$

giving the principal direction $tan2\alpha = \frac{2I_{xy}}{I_x - I_y}$. The domain is split by the principal axis according to the largest principal inertial moment.
The numerical integration is performed for a triangulation constructed from the existing point set. The existing point set has to describe the domain in a user acceptable way. Usually, triangulation of a given domain is arbitrary. Therefore, the user has to ensure the modeling of the mesh density function is appropriate with the given set of points (see also 3.2).

2.3 Coupling nodes on the subdomain boundaries

The coupling nodes on the subdomain boundaries have to be generated deterministically to avoid additional communication. The problem is twofold. First, the global namespace of the coupling nodes has to be unique. Second, the co-ordinates of the nodes generated on the boundary between two or more subdomains have to be the same to ensure conforming meshes.
The subdomains are recursively generated in parallel. Initially, the problem is loaded onto all processors. All the processors perform the first split. Processors $0, \ldots, p/2-1$ are assigned the first part of the domain and processors $p/2, \ldots, p-1$ are assigned the second part of the domain. Assignment is performed straightforward by the identity mapping, namely subdomain 1 is mapped onto processor 1, subdomain 2 is mapped onto processor 2 and so on.
Resulting coupling nodes on subdomain boundaries are generated as described below in section 2.4 and are numbered consecutively. The generation of boundary nodes is deterministic. The same algorithm is performed on every processor. Therefore the available information on the co-ordinates of the coupling nodes is the same on every processor. Therefore, no communication is needed after the split.
In the second and the following recursion levels first the number of newly generated nodes is counted in every group of processors performing the same split, second the total number of coupling nodes is exchanged between the interested processor groups and third the globally unique names are assigned to the coupling nodes. The collective communication (`MPI_Allreduce`) synchronises the processor groups.

The mesh generation is performed sequentially on every processor without further communication on the last recursion level. The implemented mesh generation algorithm is decribed in [24].

2.4 To guarantee conform meshing on subdomain boundaries

Based on the input data we are using a technique defining piecewise node placements on linear segments. The segment is defined by two nodes with prescribed mesh size values f_1 and f_n and a connecting straight line. The node with the smaller mesh size value f_1 is assumed to be the starting point of the recursive node generation process illustrated in Figure 3.

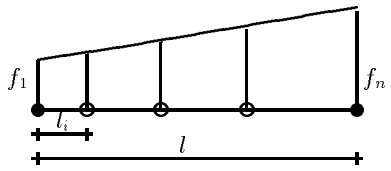

Fig. 3. *Discretisation of a line segment*

The distances of the $n-1$ intermediate nodes on the segment are computed by the recursion

$$\begin{aligned} l_1 &= \beta f_1 \\ l_i &= l_{i-1} + \beta \sum_{j=1}^{i} j \end{aligned} \tag{3}$$

whereby β is computed in a preprocessing step.
Following [19] the length of the line segment to discretize L is calculated from the co-ordinates x_1, y_1 and x_n, y_n of the starting point and the endpoint, respectively. The mean value of the expected element sizes in the starting and end point $\bar{f} = \frac{1}{2}(f_1 + f_n)$ is distributed linearly onto the number of intermediate line segments $n = \lceil \frac{L}{\bar{f}} \rceil$ in proportion to the weighted distances giving $n(n-1)$ fractions of length

$$\Delta l = \frac{2(f_n - f_1)}{n(n-1)} \tag{4}$$

The total length of the distances is checked by

$$l_{tot} = n f_1 + \Delta l \sum_{i=1}^{n} i(n-i). \tag{5}$$

and β is determined by $\beta = \frac{\Delta l}{l_{tot}}$ to scale the single distance values l_i to fit into the total segment length L.

3 Results

Results of the proposed method are compared to the traditional sequential approach of mesh generation as described in [24]. Parameters to be compared are the mesh quality, the partitioning quality and the parallel efficiency of the mesh generation. The influence of the some what different mesh on the solution of PDE will be reported in a follow up paper. Two examples are composed here with detailed result data. The first example (m1) has a simlpified geometry without mesh grading. Figure 4 shows the mesh generated on 1 and on 4 processors, respectively. The other example is a graded mesh. Figure 5 shows the mesh generated an 1 and on 8 processors, respectively.

3.1 Mesh quality

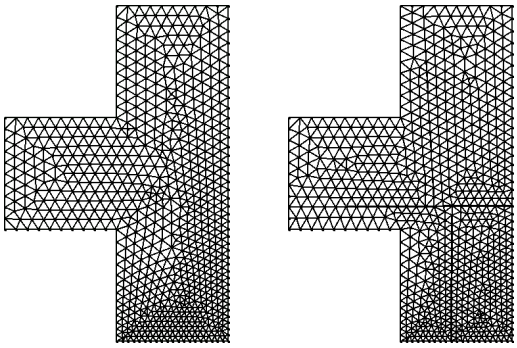

Fig. 4. *Serial and parallel mesh generation example m1*

Meshes generated by the proposed parallel generation algorithm differ from sequentially generated meshes with respect to the additional line segments of the subdomain cuts. These line segments are clearely visible in the parallel generated mesh. they are not part of the sequentially generated mesh. The main advantage of these line segments is the a-priori short cut length. Local improvement of the mesh by flipping single elements between subdomains to shorten the subdomain boundaries is not necessary.
Smoothing the element shapes in the parallel generation process is restricted to the elements of a single subdomain. Nodes can only be moved inside the subdomains. Intersubdomain smoothing is not implemented to avoid communication. Nevertheless, the comparison of the resulting minimum angle distribution for a number of triangulations shows similar results for the sequential and the parallel case, respectively (see Table 1 with results for the generation of Example m2 (compare Figure 5) on different numbers of processors).

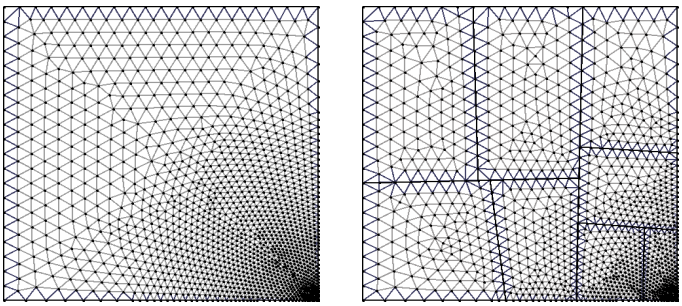

Fig. 5. *Serial and parallel mesh generation example m2*

Table 1. *Minimum angle distribution for Example m2 (Figure 5)*

Number of processors	1	2	4	8
$\bar{\alpha}$	59.73	59.70	59.70	59.66
σ	7.43	7.44	7.74	7.96

3.2 Partitioning quality

Quality of the partitioning is estimated by the following features

- load balance,
- interprocessor communication and
- network complexity or subdomain connectivity.

The attraction of the inertial method results from the perfect solution to the last two features ([11],[14]) . Inertial method is a member of the recursive bisection partitioning class. Therefore, the mapping of the resulting mesh is well investigated and the network complexity is bounded. Straight line cuts produced by the application of the inertial method to the original, unrefined subdomain geometry result in extremely short subdomain boundaries. Minimum interprocessor communication is achieved without additional smoothing and improving a preliminary partitioning like in the sequential approach based on exisiting meshes.
The load balance of the resulting partitioning is more critical. Load balance is achieved by the minimum mean difference from an average load per subdomain in the least square sense. The known sequential implementations of the mesh based inertial method guarantee a perfect result.
For the examples shown in Figures 4 and 5 the resulting mean difference from average load are shown in Table 2. The quality of the proposed method depends on three characteristic features

Table 2. *Mean difference from average load 4*

Number of processors	Example m1	Example m2
2	12.18%	4.12%
4	13.01%	4.56%
8	6.60%	3.45%

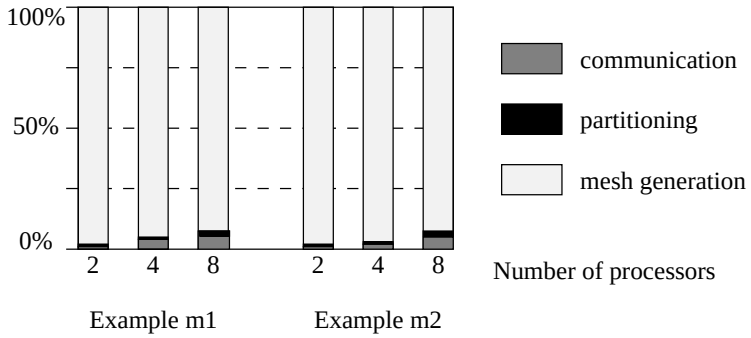

Fig. 6. *Runtime distribution for parallel mesh generation*

- quality and appropriatness of the input data,
- quality (accuracy and smoothness) of the mesh density function and
- the number of the processors.

Errors in the estimation of the load distribution are propagated with the recursion levels. The accuracy of this estimation depends on the accuracy of the numerical integration. The numerical integration is performed for intermediate triangular meshes generated with the given point set. If the given point set does not reflect the users description of the mesh distribution or the triangular mesh is to coarse to transfer sufficient information the estimation of the mesh distribution and the load distribution will be less accurate. For a given problem the loss in accuracy grows with the number of processors.

3.3 Parallel efficiency

Parallel efficiency depends on the time complexity of the mesh generator itself and on the parallel overhead due to additional or not parallelisable parts of the algorithm. Parallelisation overhead introduced by additional communication is not significant for the proposed method. All the communications needed are performed completely synchronised and can be implemented as collective communications of type `ALLREDUCE` within processor groups (compare details of

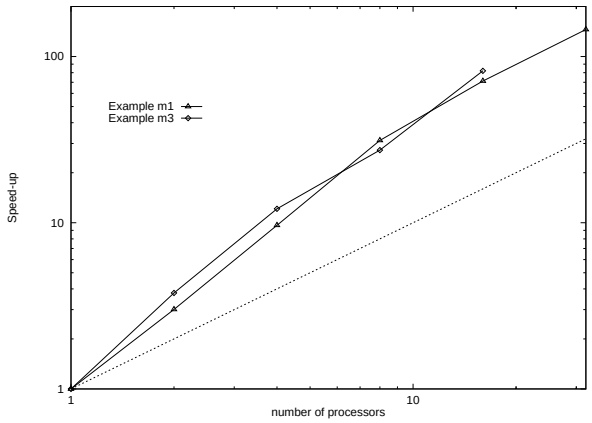

Fig. 7. *Speed-up of parallel mesh generation*

MPI implementations in [12]). These communications are known to be implemented in logarithmic time very efficiently on a large number of multiprocessor architectures (see [27], [4]).
The mesh generation process is perfectly parallel. Therefore, the application scales very well. Due to the overlinear complexity of the meshing algorithm the speedup is superlinear (see Figure 7). Runtime analysis on a Pyrsytec PowerXPlorer-32 (MPC 601 with Parix 1.3 and gcc2.7.1) shows the minimal impact of the additional parallelisation overhead as shown in Figure 6.
The ratio between the time for the mesh generation and the time spent on communication determines the efficiency of the parallel approach. The time for the communication is bound by the number of messages to be exchanged. The number of messages to be exchanged depends on the number of processors, only. It is independent on the problem size. Therefore, the efficiency of the proposed method can be tuned by scaling up the problem size - increasing the number of elements to be generated on every subdomain - for a given processor configuration.

4 Summary

The method presented generates adaptive meshes in parallel. Input data origin from CAD description. The partitioning is based on the geometric description. Triangulation in advance is not necessary. The mesh generation process works on the partitioned geometric description.
Results are comprehensive and promising. Mesh quality is ensured in the parallel approach. The quality of the partitioning is comparable to the mesh based methods. Addional improvement by local heuristics, flipping of elements between subdomains to minimise the interprocessor communication and optimise subdomain shapes is not necessary. The parallel overhead can be neglected. The algorithm is scalable.

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